COMMENTS OVER "SOME NON-CHROMATIC RINGS"

MAHESH N. DUMALDAR & PRAKASH K. SHARMA
School Of Mathematics
Vigyan Bhawan, Khandwa Road,
INDORE [M.P.] - 452 001 [INDIA]

ABSTRACT
In [3], a class of non-chromatic finite local rings 
(R,\mathfrak{m}) is given. It is proved that if R has prime 
residue field \mathbb{F} and \( p \geq 3 \), then 
\( p^{2+2} \leq \chi(R) \leq p^2 + (\frac{p+1}{2}) \), moreover, 
\( \chi(R) = p^2 + 2 \) for \( p = 2, 3, 5 \). Here we 
have shown that, in general, \( \chi(R) \neq p^2 + 2 \) by showing that 
\( \chi(R) = p^2 + 3 \) for \( p = 7 \).

INTRODUCTION
Let (R,\mathfrak{m}) be a finite, commutative local ring with 
identity such that (i) \( \mathfrak{m}^3 = 0 \) (ii) \( \dim_{\mathbb{F}} \frac{\mathfrak{m}}{\mathfrak{m}^2} = 3 \) (iii) 
\( \dim_{\mathbb{F}} \mathfrak{m}^2 = 1 \) and (iv) \( \text{Ann}\mathfrak{m} = \mathfrak{m}^2 \), where \( \mathbb{F} = \frac{R}{\mathfrak{m}} \), the 
residue field of R.

* Corresponding author.
Consider a simple graph with vertices elements of \( R \) such that any two distinct elements \( x, y \) of \( R \) are adjacent if and only if \( xy = 0 \). We denote by \( \chi(R) \), the chromatic number of \( \text{Graph}_R \), the minimal number of colours required to colour the vertices of the \( \text{Graph}_R \) such that no two adjacent vertices have same colour.

Throughout the article \( R \) shall denote a finite local ring with prime residue field \( \mathbb{F}_p \) (\( p \): Characteristic of \( \mathbb{F} \), \( p \geq 3 \), as described above. We shall freely use the terminology of [1] and [2] in this article.

This is proved in [3] that for all \( p \geq 3 \), \( p + 2 \leq \chi(R) \leq p^2 + \left( \frac{p+1}{2} \right) \). Further, the calculations of \( \chi(R) \) for \( p = 2, 3, 5 \), give that \( \chi(R) = p + 2 \). In this article we shall show that for \( p = 7, \chi(R) = p^2 + 3 \). Thus, not for all \( p, \chi(R) = p^2 + 2 \).

\[ \phi 1 \text{ PRELIMINARIES} \]

We shall prove here three technical lemmas. To start with, we collect some definitions and results from [3] for use in this article.

Definition 1.1 [3, Definition 2.1]: Let \( x \in \mathbb{M}^2 \), \( x^2 = 0 \). We denote by \( [x] \) the clique \( Rx + \mathbb{M}^2 \) i.e.,

\[ [x] = \{ \lambda x + k | \lambda \in R, k \in \mathbb{M}^2 \} \]

Note: (i) By different (distinct) cliques \( [x] \) and \( [y] \) we shall always mean \( [x] \neq [y] \).

(ii) \( |[x]| = p^2 \).
Definition 1.2 [3, Definition 2.2]: An element \( w \in \mathfrak{m} \) is called a connector if and only if

(i) There exist \( x, y \in \mathfrak{m} - \mathfrak{m}^2 \), such that \( x \not\in \mathfrak{m}y^2 \), \( x^2 = y^2 = 0 \), and \( w \in \text{Ann}(x) \cap \text{Ann}(y) \).

(ii) \( w^2 \neq 0 \)

Note: By different connectors we shall always mean connectors corresponding to different pairs of cliques.

Remark: \( \text{Ann}(w) = (x, y) + \mathfrak{m}^2 \).

Lemma 1.3 [3, Lemma 2.5]: Let \( x \in \mathfrak{m} - \mathfrak{m}^2 \), \( x^2 = 0 \) and let \( w \in \text{Ann}(x) \) be such that \( w^2 \neq 0 \). Then there exists \( y \in \mathfrak{m} \) such that \( y^2 = yw = 0 \) and \( \mathfrak{m} = (x, y, w) \).

Lemma 1.4: Let \((R, \mathfrak{m})\) be a local ring with prime residue field \( \mathbb{F}_p \). Then \( \mathfrak{m} = (x, y, w) \) where \( x^2 = y^2 = xw = yw = 0 \), and \( w^2 = xy \neq 0 \). Further \([x], [y], [x - \frac{t}{2}y + tw] \); \( t = 1, 2, \ldots, p-1 \) are all distinct cliques in Graph\( R \).

Proof: This follows from [3, Lemma 2.7 and Lemma 2.5]

Remark 1.5 [3, Remark (ii), Lemma 2.7]: Let \( w \) be a connector. Then each element of \( \mathfrak{m}^2 \) is also a connector to the same pair of cliques. Further, \((t_1 w + k)(t_2 w + 1) \neq 0 \) for any \( t_1, t_2 \in \mathbb{F}_p^\times \), \( k, l \in \mathfrak{m}^2 \), and every element adjacent to \( w \) is also adjacent to each element of \( \mathfrak{m}^2 \) and conversely. Therefore while colouring the Graph\( R \), we can assign a single colour to the elements in the set \( \mathfrak{m}^2 \). Hence while colouring the connectors, we have to deal with only \( p + 1 \) different connectors as representatives. Hereafter, while considering the subgraph of connectors we shall
consider some \( p+1 \) \( C_2 \) connectors which are pairwise linearly independent \( \text{mod} \ M^2 \). This subgraph will be called the 'Graph Of Connectors'.

Theorem 1.6 [3, Theorem 3.1]: For all \( p \geq 3 \), we have
\[
2p + 2 \leq \chi(R) \leq p^2 + \left( \frac{p+1}{2} \right)
\]

Observation 1.7 [3, Observation 0.1]: Let \( x \in M \cdot M^2 \), \( x^2 = 0 \). Let \( w_1, w_2 \in \text{Ann}(x) - [x] \). Then \( w_1^2 \neq 0 \) for \( i = 1, 2 \).

Further \( w_1 w_2 \neq 0 \), since otherwise \( \{x \cup \{w_1, w_2\}\} \) will be a clique with \( p^2 + 2 \) elements. Therefore all such \( w \)'s can be given one colour. If, however, the elements of \( \text{Ann}(x) - [x] \) are coloured by three different colours, then as to colour \([x]\) we need additional \( p^2 \) colours, this colouring of \( \text{Graph}\ R \) shall require at least \( p^2 + 3 \) colours. Thus if \( \chi(R) = p^2 + 2 \), to colour elements of \( \text{Ann}(x) - [x] \) we can use at most two colours.

Lemma 1.8: Let \([x], [y], [z]\) be three different cliques in \( \text{Graph}\ R \), and let \( w \) be connector of the cliques \([x], [y]\). Then there exists a unique clique \([q]\) and a connector \( w_1 \) of \([z]\) and \([q]\) adjacent to \( w \).

Proof: By the Lemma 1.4, \( M = (x, y, w) \) where \( x^2 = y^2 = xw = yw = 0 \), \( w^2 = xy \neq 0 \). Further, there exists \( t \in \mathbb{F}_p^2 \) such that \([z] = [x, y + tw]\). If for a clique \([q]\) the connector \( w_1 \) of \([z]\) and \([q]\) is adjacent to \( w \), then
\[
\begin{align*}
w_1 &\in \text{Ann} w = (x, y) + M^2 \\
&\implies w_1 = t_1 x + t_2 y + k \quad \text{where} \quad t_1, t_2 \in \mathbb{F}_p, \ k \in M^2 \\
&\implies w_1^2 = 2t_1 t_2 xy \neq 0
\end{align*}
\]
\[ \Rightarrow t_1, t_2 \in F_p^* \]

Thus we can take \( w_1 = x + t_3y \) for \( t_3 \in F_p^* \). As

\[ w_1(x - \frac{t}{2}y + tw) = 0 \]

\[ \Rightarrow (x + t_3y)(x - \frac{t}{2}y + tw) = 0 \]

\[ \Rightarrow t_3 = \frac{t}{2} \]

Hence, for cliques \([x - \frac{t}{2}y + tw]\) and \([x - \frac{t}{2}y - tw]\), \( w_1 = x + \frac{t}{2}y \) is a connector and is adjacent to \( w \). Further, if for \( s \in F_p^* \), \( w_1 \) is adjacent to \([x - \frac{s}{2}y + sw]\), then

\[ (x + \frac{t}{2}y)(x - \frac{s}{2}y + sw) = 0 \]

\[ \Rightarrow \frac{s}{2} = \frac{t}{2} \]

\[ \Rightarrow s = t \]

Hence the result follows.

Lemma 1.9: Let \([x_i]\), \([x_j]\) and \([x_k]\), \([x_l]\) be two pairs of different cliques. If \( w \) is connector of \([x_i]\) and \([x_j]\) and also of \([x_k]\) and \([x_l]\) then

\[ [x_i] = [x_k] \text{ or } [x_l] \]

Further, if \([x_i] = [x_k]\) then \([x_j] = [x_l]\) and if \([x_i] = [x_l]\) then \([x_j] = [x_k]\).

Proof: By our assumption and the remark following Definition 1.2,

\[ \text{Ann } w = (x_i, x_j) + m^2 = (x_k, x_l) + m^2 \]

\[ \Rightarrow x_k = t_1x_i + t_2x_j + q \quad t_1, t_2 \in F_p, q \in m^2 \]

\[ \Rightarrow 2t_1t_2x_i x_j = x_k^2 = 0 \]

\[ \Rightarrow t_1 = 0 \text{ or } t_2 = 0 \]
Let \( t_2 = 0 \). Then \( x_k = t_1 x_i + q \). Thus \( [x_k] = [x_i] \). We can easily check that in this case \( [x_1] = [x_j] \). The other part of the statement will follow similarly.

**Lemma 1.10**: Let \( X = \{1, 2, \ldots, k\} \), \( k \geq 4 \), and let \( A_1, A_2, \ldots, A_m \) \((m \geq 4)\) be distinct subsets of \( X \) such that 
\[
|A_i| = 1 \text{ or } 2 \quad \text{and} \quad A_i \cap A_j \neq \emptyset \quad \text{i.e.,} \quad |A_i \cap A_j| = 1, \quad \text{for all} \quad 1 \leq i \neq j \leq m.
\]
Also \( \bigcup_{i=1}^m A_i = X \). Then
\[
\bigcap_{i=1}^m A_i = 1 \quad \text{and} \quad m = k \text{ or } k-1.
\]

**Proof**: Case I : If for some \( A_k \), \( |A_k| = 1 \) then the result is obvious and in this case \( m = k \).

Case II : Suppose for all \( A_i, 1 \leq i \leq m, |A_i| = 2 \). We have
\[
\bigcap_{i=1}^m A_i = \bigcap_{i=2}^m (A_i \cap A_1)
\]
Let \( A_1 = \{u, v\} \), \( 1 \leq u \neq v \leq k \). If \( A_1 \cap A_i = A_1 \cap A_j \), for all \( 2 \leq i \neq j \leq m \), then we are through and \( m = k-1 \).

Next, assume without loss of generality that \( A_1 \cap A_2 \neq A_1 \cap A_3 \). Then, let
\[
A_1 \cap A_2 = \{u\}, \ A_1 \cap A_3 = \{v\}
\]
\[
\implies A_2 = \{h, u\}, \ A_3 = \{l, v\} \quad (1 \leq h, l \leq k)
\]
As \( A_2 \cap A_3 \neq \emptyset \), \( h = 1 \). Thus \( A_1 = \{u, v\}, A_2 = \{h, u\}, A_3 = \{h, v\} \) where \( u, v, h \) are distinct. Further, we have \( A_4 \cap A_i \neq \emptyset, i = 1, 2, 3 \). This however is not possible unless \( |A_4| > 2 \). This contradicts the assumption that \( m = 2 \). Consequently \( \bigcap_{i=1}^m A_i = 1 \) and \( m = k-1 \).
\section{Main Results}

Theorem 2.1: If the 'Graph Of Connectors' is coloured using $k \geq \frac{p+1}{2} \geq 4$ colours, then to colour $\text{GraphR}$ we shall need $\geq p + 3$ colours.

Proof: Let $C_1, C_2, \ldots, C_{p+1}$ be all distinct cliques [Lemma 1.4] and let $w_{ij}$ ($i \neq j$) be a connector of $C_i$ and $C_j$ where $w_{ij} = w_{ji}$ for all $i \neq j$. Put

$$A_i = \{w_{i1}, w_{i2}, \ldots, w_{i,i-1}, w_{i,i+1}, \ldots, w_{i,p+1}\}$$

for $i = 1, 2, \ldots, p+1$.

We have:

a) $|A_i| = p$ for all $1 \leq i \leq p+1$.

b) $A_i \cap A_j = \{w_{ij}\}$, for all $i \neq j$. Hence $|A_i \cap A_j| = 1$ for all $1 \leq i \neq j \leq p+1$ [Lemma 1.9]

c) For any $w \in A_i$, there exists $j (\neq i)$ such that $A_i \cap A_j = \{w\}$.

d) For any two distinct sets $A_i$ and $A_j$, if $w \in A_i$ and $w \not\in A_i \cap A_j$ then by Lemma 1.8 there exists unique $w' \in A_j$ such that $ww' = 0$. Thus for each element in $A_i$ which is not in $A_i \cap A_j$, there is exactly one element in $A_j$ which is adjacent to this element. Hence there are $p-1$ pairs $(w, w') \in A_i \times A_j$ such that $ww' = 0$.

Consider a colouring of $\text{GraphR}$. Let $K_i$ be the set of all colours used to colour the elements of $A_i$. Now, suppose the 'Graph Of Connectors' is coloured using $k \geq \frac{p+1}{2} \geq 4$ colours, say $f_1, f_2, \ldots, f_k$ and $\text{GraphR}$ is coloured with $p + 2$ colours. Then $|K_i| = 1$ or 2 [Observation 1.7] for all $1 \leq i \leq p+1$, and if $K_i \neq K_j$.
\[ |K_1 \cap K_j| = 1. \] Further \[ | \bigcup_{i=1}^{p+1} K_i | = k. \] Now, let the number of distinct \( K_i \)'s are more than or equal to 4. Then, by Lemma 1.10 \[ | \bigcap_{i=1}^{p+1} K_i | = 1. \] Without loss of generality, we can assume that \( \bigcap_{i=1}^{p+1} K_i = \{f_1\} \). Hence we have the following:

e) For any \( i \), \( K_i = \{f_1\} \) or \( \{f_1, f_s\} \) for some \( 2 \leq s \leq k \).

f) As \[ \bigcup_{i=1}^{p+1} K_i = \{f_1, f_2, \ldots, f_k\}, \] for any \( 2 \leq s \leq k \), there exists \( 1 \leq i \leq p+1 \) such that \( K_i = \{f_1, f_s\} \).

g) By f), one of the elements in \( A_i \) has colour \( f_s \).

Hence by c) there exists \( A_j, 1 \leq j (\neq i) \leq p+1 \) such that

\[ K_j = K_i \]

We, now, have two possibilities:

**CASE I**: \( 2(k-1) = p+1 \) i.e., \( k = \frac{p+1}{2} + 1 \)

In this case, for each \( \{f_1, f_s\}, 2 \leq s \leq k \), we have exactly two indices \( i, j, 1 \leq i \neq j \leq p+1 \) such that \( K_i = K_j = \{f_1, f_s\} \).

**CLAIM**: If \( \{w\} = A_i \cap A_j \), then \( w \) is the only element of \( A_i \cup A_j \) which has colour \( f_s \).

Let for some \( w' \) in \( A_i \), \( w' \not\in A_i \cap A_j \) the colour of \( w' \) be \( f_s \). By c), there exists \( h \neq i, h \neq j \) such that \( A_i \cap A_h = \{w'\} \). Then \( K_h = \{f_1, f_s\} \). This contradicts the fact that \( 2(k-1) = p+1 \). Hence the claim follows.
By the claim, all elements in \((A_i \cup A_j) - (A_i \cap A_j)\) have colour \(f_1\). This is not possible by d), otherwise adjacent elements will have same colours. Thus 2(k-1) = p+1 is not possible.

CASE II : \(2(k-1) < p+1\) i.e., \(k = \frac{p+1}{2}\) (as \(k \geq \frac{p+1}{2}\)).

In this case, \(2(k-1) = p-1\). Thus two more \(K_i's\) (say) \(K_p\) and \(K_{p+1}\) remain to be determined. Here we have two possibilities:

(i) Let \(K = \{f_1\}\).

In this case, by d), \(K_{p+1} \neq \{f_1\}\). Hence \(K_p + 1 = \{f_1, f_s\}\)
for some \(2 \leq s \leq k\). As \(k \geq 4\), there is some \(q, 2 \leq q \neq s \leq k\) such that
\[K_m = K_n = \{f_1, f_q\}\]
for \(1 \leq m \neq n \leq p-1\)
and
\[K_i \neq \{f_1, f_q\}\]
for any \(i \neq m, i \neq n\).

i.e., the combination \(\{f_1, f_q\}\) appears exactly twice.

Now, by the argument in the CASE I, we get contradiction. Similarly we proceed if \(K_p + 1 = \{f_1\}\).

(ii) \(|K_p| = |K_{p+1}| = 2\)

Here, either \(K_p \neq K_{p+1}\) or \(K = K_{p+1}\). Thus as \(k \geq 4\), we again conclude that at least one combination \(\{f_1, f_q\}\)
appears exactly twice. Thus, as above, \(2(k-1) < p+1\) is not possible.

Further, let the 'Graph Of Connectors' is coloured using \(k > \frac{p+1}{2} + 1\) colours. Then by a simple counting argument we conclude that, \(|K_i| \geq 3\) for at least one i. Next, if the number of distinct \(K_i's\) are less than four, then similar arguments will show that
there exists a $K_i$ with $|K_i| \geq 3$. Therefore, in any case, there exists a $K_i$ such that $|K_i| \geq 3$. Now, as to colour the corresponding clique $C_i$ we require another set of $p^2$ colours, to colour $\text{Graph}_R$ we shall need $\geq p^2+3$ colours. Hence the result follows.

Theorem 2.2 : Let $(R, \mathfrak{M})$ be finite local ring with prime residue field $\mathbb{F}_p$, where $p = 7$. Then $\chi(R) = p^2 + 3$.

Proof : By the Lemma 1.3, there exist $x, y, w \in R$ such that $\mathfrak{M} = (x, y, w)$, where $x^2 = y^2 = xw = yw = 0$, $w = xy \neq 0$. Further, by Lemma 1.4 there are eight different cliques,

$$
C_1 = [x], C_2 = [y], C_3 = [x+3y+w],
$$
$$
C_4 = [x+3y+6w], C_5 = [x+5y+2w], C_6 = [x+5y+5w],
$$
$$
C_7 = [x+6y+3w], C_8 = [x+6y+4w].
$$

in $\text{Graph}_R$. One can easily calculate that the 'Graph Of Connectors' has 28 vertices. We can take these the following:

1) $w$ 2) $x+w$ 3) $x+2w$ 4) $x+3w$ 5) $x+4w$ 6) $x+5w$ 7) $x+6w$
8) $y+w$ 9) $y+2w$ 10) $y+3w$ 11) $y+4w$ 12) $y+5w$ 13) $y+6w$
14) $x+y$ 15) $x+2y$ 16) $x+4y$ 17) $x+y+3w$ 18) $x+y+4w$
19) $x+2y+2w$ 20) $x+2y+5w$ 21) $x+3y+3w$ 22) $x+3y+4w$
23) $x+4y+w$ 24) $x+4y+6w$ 25) $x+5y+w$ 26) $x+5y+6w$
27) $x+6y+2w$ 28) $x+6y+5w$.

We give below the Table 1 of connectors where the $(i,j)$\textsuperscript{th} element of the table is the connector of the cliques $C_i$ and $C_j$, and is denoted by the corresponding number of the connector given above.
### TABLE 1

<table>
<thead>
<tr>
<th>CLIQUES</th>
<th>C₁</th>
<th>C₂</th>
<th>C₃</th>
<th>C₄</th>
<th>C₅</th>
<th>C₆</th>
<th>C₇</th>
<th>C₈</th>
</tr>
</thead>
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<td>1</td>
<td>5</td>
<td>4</td>
<td>2</td>
<td>7</td>
<td>6</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>C₂</td>
<td>1</td>
<td>13</td>
<td>8</td>
<td>10</td>
<td>11</td>
<td>9</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>C₃</td>
<td>5</td>
<td>13</td>
<td>16</td>
<td>28</td>
<td>17</td>
<td>19</td>
<td>26</td>
<td></td>
</tr>
<tr>
<td>C₄</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>18</td>
<td>27</td>
<td>25</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>C₅</td>
<td>2</td>
<td>10</td>
<td>28</td>
<td>18</td>
<td>15</td>
<td>24</td>
<td>21</td>
<td></td>
</tr>
<tr>
<td>C₆</td>
<td>7</td>
<td>11</td>
<td>17</td>
<td>27</td>
<td>15</td>
<td>22</td>
<td>23</td>
<td></td>
</tr>
<tr>
<td>C₇</td>
<td>6</td>
<td>9</td>
<td>19</td>
<td>25</td>
<td>24</td>
<td>22</td>
<td>14</td>
<td></td>
</tr>
<tr>
<td>C₈</td>
<td>3</td>
<td>12</td>
<td>26</td>
<td>20</td>
<td>21</td>
<td>23</td>
<td>14</td>
<td></td>
</tr>
</tbody>
</table>

The adjacency status in the 'Graph Of Connectors' is as below:

### TABLE 2

<table>
<thead>
<tr>
<th>VERTEX</th>
<th>ADJACENT VERTICES</th>
<th>VERTEX</th>
<th>ADJACENT VERTICES</th>
<th>VERTEX</th>
<th>ADJACENT VERTICES</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>14, 15, 16</td>
<td>11</td>
<td>6, 20, 28</td>
<td>21</td>
<td>7, 9, 16</td>
</tr>
<tr>
<td>2</td>
<td>13, 20, 22</td>
<td>12</td>
<td>5, 18, 22</td>
<td>22</td>
<td>2, 12, 16</td>
</tr>
<tr>
<td>3</td>
<td>10, 17, 25</td>
<td>13</td>
<td>2, 23, 25</td>
<td>23</td>
<td>4, 13, 24</td>
</tr>
<tr>
<td>4</td>
<td>9, 23, 28</td>
<td>14</td>
<td>1, 27, 28</td>
<td>24</td>
<td>5, 8, 23</td>
</tr>
<tr>
<td>5</td>
<td>12, 24, 27</td>
<td>15</td>
<td>1, 25, 26</td>
<td>25</td>
<td>3, 13, 15</td>
</tr>
<tr>
<td>6</td>
<td>11, 18, 26</td>
<td>16</td>
<td>1, 21, 22</td>
<td>26</td>
<td>6, 8, 15</td>
</tr>
</tbody>
</table>

continued...
It is easy to see that the chromatic number of the 'Graph Of Connectors' is 3. Hence from the proof of [3, Theorem 3.1] it follows that $\chi(R) \leq p^2 + 3$. By Theorem 2.1, if the 'Graph Of Connectors' is coloured using $k \geq 4$ colours then $\text{GraphR}$ requires at least $p^2 + 3$ colours. Now, it remains to find the minimal number of colours needed to colour $\text{GraphR}$ when 'Graph Of Connectors' is coloured using 3 colours.

We know by observation 1.7 that if $\text{GraphR}$ is coloured using $p^2 + 2$ colours then each row (or column) in the Table 1 shall be coloured using at most two colours. We find out all possible ways of colouring of the 'Graph Of Connectors' using three colours and for each of the colouring we check the observation 1.7. This is done using a computer programme† formulated with the help of Algorithms 7.7 and 7.8 in [5].

† The programme is available on request.
On implementing the programme we get that there does not exist any such colouring i.e., there exists no colouring of the 'Graph Of Connectors' by three colours which satisfies observation 1.7. Therefore \( \chi(R) \geq p^2 + 3 \), and hence \( \chi(R) = p^2 + 3 \).

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